A New Results on Fenchel Duality Theorem in Frechet Spaces

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Abstract

The Fenchel Duality Theorem in Frechet Spaces can be developed at many levels and it has a wide variety of important applications to Functional Analysis and Graph Theory .The space B on which the operators acts may be an ordered as Banach Space on the Frechet space.

Keywords

Convex function , Frechet spaces, Frechel duality

Introduction

Kothe and Robertson [1, 10] constructed some example of Fenchel Duality Theorem in Frechet Spaces that Let X and Y be real locally convex space and X^* and Y^* denote their respective topologies dual. For $M: X \to R$ U { ∞} ,the Legendre-Fenchel transform of M ,$M^*: X^* \to R\ U\ \{\infty\}$, is defined by

$$M^*\ f := \operatorname*{Sup}_{x\in X}[fx - Mx](f \in X^*)$$

With $N: Y \ \to R\ \ U\ \{\infty\}$ and $A: X \to Y$ linear and continuous , it is of fundamental importance in convex optimization to know when we can say

$\inf_{x\in X}[Mx + N\ Ax] + \min_{g\in Y}[M^*(\ -A'g) + N^*$g= 0] ....(1)

For example, characterizes a solution to the minimum problem

$\inf_{x\in X} Mx$ subset to $Ax \in c$

Where $B \subset X$ and $C \subset Y$ are non-empty.Here $M: B \ \to R\ \ U\ \{\infty\}$ and we define $M: X \ \to R\ \ U\ \{\infty\}$ by setting $Mx := \infty$ for $x \notin B$ and N= I_c where $I_c: Y \ \to R\ \ U\ \{\infty\}$ is the indicator function of C which is defined by $I_c(y) = 0\ \ (y \in C)$ and $I_c(y) := \infty(y \notin C)$.

The Fenchel duality theorem states that the equality

$\inf_{x\in X}[Mx + Nx] + \min_{f\in x}[M^*\ f + N^*(f)$= 0].......................................(2)

holds if $X = R^n$ and if M and N are convex such that there is a point in $D(M) \cap D(N)$ at which M or N is continuous . Here D(M) denotes the effective domain of the function $M: X \rightarrow R^n U \{\infty\}$, and is defined by D(M)= $\{x|x \in X, Mx \in R\}$. Also we say M is proper if $D(M) \neq 0$.

Definition(1.1)

Let M,$N: X \rightarrow R^n U\{\infty\}$ be convex function .The inf-convoltuion of M with N at $x \in X$,written $(M \nabla N)(x)$, is defined by

$(M \nabla N)\underset{x \in X}{(x)} = \inf[M(x-y) + Ny]\ (x \in X)$..................(3)

We say that the inf-convolution is exact if, for each $x \in X$,there exist some $y_0 \in X$ such that

$$(M \nabla N)(x) = M(x - y_0) + Ny_0 + Ny_0$$

It is clear , therefore ,that if we show

$(M + N)^* f$= $(M^* \nabla N^*)(f)\ (f \in N^*)$..(4)

And if the inf- convolution in the right hand side of (4) is exact ,then the conclusion n the Fenchel theorem (2) follows by taking $f = 0$. Then inf-convolution theorem states that (4) holds under the conditions of the Fenchel theorem.

In proving our main results we use the following

Proposition(1.1)

Let X be a barraled space and M ,N:X→ $R\ U\ \{\infty\}$ be convex ,proper and lower semicontinuous such that D(M) ∩ int D(N)≠ 0.

Then

$\inf_{x \in X}[Mx + Nx] + \min_{f \in X^*}[M^* f + N^*(-f)$= 0] ....(5)

Proposition (1.2)

If $\bigcup_{\lambda > 0} \lambda(D(L) + B(X)) = Z$

Then for each $U \in \mathcal{U}$ and λ∈ R ,$Q_{U,\lambda} = \{g|g \in Z^*, L^*g \leq \lambda, B^t g \in U^0\}$.........................(7)

Is either empty or $\sigma(Z^*, Z) - compact$.

Proof

Let X be Frechet space and Z be fully barrelled and $X^*$and Z^*their respective dual spaces. Let $\mathcal{U}$ be a neighbourhood base at the origin in X consisting of convex ,balanced and absorbing sets. For each $U \subset \mathcal{U}$,let $U^0 \subset X^*$ denote its polar. Let $B: Z \rightarrow RU\{\infty\}$ be convex,proper and lower semicontinuous and $B: X \rightarrow Z$ be linear and continuous.

For each $\in Z$, there exists $\lambda_0 > 0$, $Z_0 \in D(L)$ and $z_0 \in D(L)$ and $x_0 \in X$ such that $z = \lambda_0 Z_0 + Bx_0$. Since $U \in \mathcal{U}$ is absorbing, there exist $\mu_0 > 0$ such that $\mu_0 x_0 \in U$. Hence ,for each $g \in Q_{U,\lambda}$.

$< z, g >= \lambda_0 < z_0, g > + < Bx_0, g > \ \leqq \lambda_0[Lz_0 + L^* g] + < x_0, B^t g >$

$\leqq \quad \lambda_0[Lz_0 + \lambda] + \mu_0^{-1} < \mu_0 x_0, B^t g >$

$\leqq \lambda_0[Lz_0 + \lambda] + \mu_0^{-1}$

It follows that $Q_{U,\lambda}$ $is\ \sigma(Z^*, Z) - bounded\ and\ hence$ equi continuous by the

Uniform Boundedness Principle.$Q_{U,\lambda}$ is therefore relatively $\sigma(z^*, z) -\ is\ compact$.

Since $\sigma(z^*, z) - is\ closed$, $Q_{U,\lambda}\ is\ \ in\ \ fact$, $\sigma(Z^*, Z) - closed$ and the result follows

Main Result

If Z= $\bigcup_{\lambda>0} \lambda(D(L) + B\ (X)$ is a closed linear subspace of Z .... (8)

Then $\forall\ f \in D(\ LB)^*$

$(\ LB)^* f = min\ (L^* g | g \in Z^*, B^t g = f\}$ .... (9)

Proof Step I

We first show to hold under assumption. L.B is clearly convex and lower semi continuous. By assumptions , there exist $z_0 \in D(L)$ and $x_0 \in X$ such that $0 = z_0\ + Bx_0$.

Hence L B (-x_0)=L(Bx_0) $= L(z_0) < \infty$ and LB is proper.It follows from this that D$(LB)^*$

non-empty.

For $f \in D(LB)^*$ define $J: X^* \rightarrow R\ U\ \{\pm\infty\}$ by

$$Jf = inf\{L^* g | g \in Z^*, B^t g = f\}$$

$$(f \in D(LB)^*$$

Since there exists a $U \in \mathcal{U}$ such that $f \in U^0$ we find that

$\{g | g \in Z^*\}\ B' g = f\} \subset \{g | g \in Z^*, B^t g \in U^0$

From this it follows that, for each λ∈ R,

$\{g | g \in \{Z^*\}\ L\ ^* g \leq \lambda, B^t g \subset Q_{U,\lambda}\}$

Where $Q_{U,\lambda}$ is defined as in .Since $B^t: Z^* \rightarrow X^*$ is continuous between the $weak^*$- topologies and L^* is $\sigma(Z^*, Z)$ –lower semicontinuous, we find using propositions that for each $\lambda \in R$,

$g | g \in \{Z^*\}\ L\ ^* g \leq \lambda, B^t g = f\}$ is either empty or $\sigma(Z^*, Z)$-compact .This allows us to replace "inf" by "min" in the definition of J, where now $J: X^* \rightarrow R\ U\ \{\infty\}$.

Next for each $h \in D(L^*)$

$J(B^1 h) = min\ \{L^* g | g \in Z^*, B^t g = B^t h\} \leqq L^* h < \infty$, which shows that J is proper,

It is easily verified that J is also convex.

We now show that J is $\sigma(X^*, X)$ −lower semi continuous by proving that for each $\lambda \in R$, the sublevel set $S(J;\lambda) = \{f|f \in X^*, Jf \le \lambda$, is $\sigma(X^*, X)$ closed. Indeed , for each $U \in \mathcal{U}$, we find that $S(J;\lambda) = \cap\, U^0 = f|f \in X^*, Jf \le \lambda, f \in U^0 = B^t(Q_{U,\lambda})$ where $Q_{U,\lambda}$ is $\sigma(Z^*, Z)$ −compact . By the $\sigma(Z^*, Z) - \sigma(X^*, X)$ continuity of B^t and the KREIN-SMULIAN theorem it then follows that S(J;λ) is, in fact, $\sigma(X^*, X) - closed$.

By the Fenchel-Mreau theorem applied to J , we conclude that $J = J^{**}$ now follows since ,for each $x \in X$,

$$LBx = L^{**}(Bx) = \sup_{y \in z^*}[< Bx, g > -L^*g]$$

$$= \sup_{g\in z^*, f \in x^*, B^t g = f}[< x, f > - \inf_{g \in z^*, B^t g = f} L^*g]$$

$$= J^*x$$

And so that ,for each $f \in D(LB)^*$

$(LB)^*f = J^{**}f = Jf = min\,\{L^*g|g \in Z^*, B^t g = f\}$

Step II

We can now extend to hold under in a simple and direct way. We achieve this extension as follows :By assumption since $0 \in B(x), given\ x \in D(L)$ we can find $\lambda_0 > 0$,$y \in D(L)$ and $z \in B(x). Such\ that\ -x = \lambda_0(y + z)$. it follows

$$0 = (1 + \lambda_0)^{-1}x + \lambda_0(1 + \lambda_0)^{-1}(y + z) \in D(L) + B(x)$$

And hence that $0 \in D(L) \cap B(x)$ by a suitable translation. We deduce from this that D (L),

B(X) ⊂ Ȥ .

Next let $i =$ Ȥ $\to Z$ be the canonical injection map with i^t:$Z^* \to$ Ȥ* denoting its (surjective) adjoint and define ζ=L o i: $\mathcal{L} \to RU$ {∞}.

Since clearly $\bigcup_{\lambda>0} \lambda(D(\mathcal{L}) + B(X)) = \bigcup_{\lambda>0} \lambda(D(L) + B(X) = \mathcal{L}$

Step I with Z replaced by $\mathcal{L}$ and L by ζ ,gives ∀ $f \in D(\mathcal{L}B)^*$

$$(\mathcal{L}B)^*f = min\,\{\mathcal{L}^*h|h \in \mathcal{L}^*, B^t h = f\}$$

It is now easy to see that $\forall f \in D(LB)^*$

$(LB)^*f = min\,\mathcal{L}^*g|$g∈ $z^*, B^t g = f$

Indeed by using D(L) ,B(X)⊂$\mathcal{L}$ we find that $(LB)^*f = (LB)^*$

And min $\{\mathcal{L}^*h|h \in \mathcal{L}^*, B^t h = f$

=min $\{\mathcal{L}^*(ig)|g \in z^*, B^t(i^t g = f\}$

=min { $L^{*}g | g \in Z^{*}, B^{t}g = f$

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